Lecture 2. Conditioning & Bayes' Rule

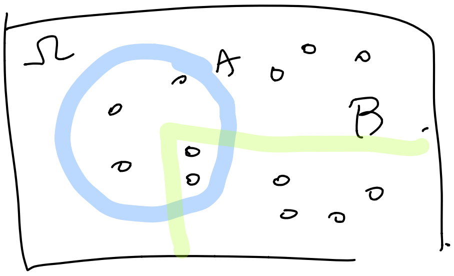
2.1 Lecture Overview

"New knowledge cause our "beliefs" to change ; original probabilities must be replaced to take new information into account."

2.2 Conditional Probabilities

* The idea of conditioning : Use new info to revise a model

Example Assume 12 equally likely outcomes.



$P(A) = 5/12$, $P(B) = 6/12$.

But if told B occurred,

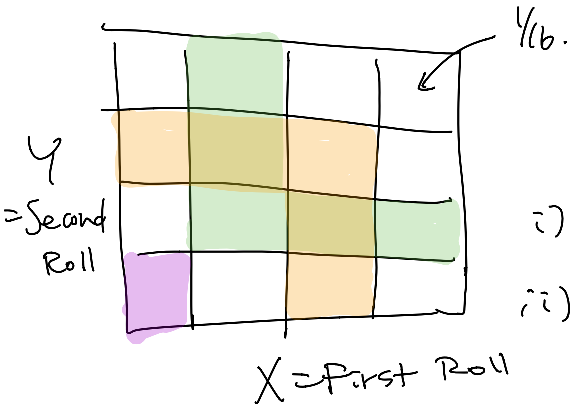
$P(A|B) = 2/6$, $P(B|B) = 6/6$.

Def Conditional Probability.

$P(A|B) =$ "Probability of A, given that B occurred"

$= P(A \cap B) / P(B)$ (defined only when $P(B) > 0$).

2.3 A Die Roll Example.



Let B be the event : $\min(X,Y) = 2$.

Let $M = \max(X,Y)$.

i) $P(M=1 \mid B) = 0$.

ii) $P(M=3 \mid B) = \dfrac{P(M=3 \text{ and } B)}{P(B)} = \dfrac{2/16}{5/16} = \dfrac{2}{5}$

2.4 Conditional Probabilities obey the same axioms.

i) $P(A|B) \geq 0$. (assuming $P(B) > 0$).

ii) $P(\Omega|B) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1.$

iii) $P(B|B) = \dfrac{P(B \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1.$ "B as the new sample space"

iv) if $A \cap C = \emptyset$, then $P(A \cup C | B) = P(A|C) + P(B|C)$.

$$= \frac{P((A \cup C) \cap B)}{P(B)} = \frac{P[(A \cap B) \cup (C \cap B)]}{P(B)} = \frac{P(A \cap B) + P(C \cap B)}{P(B)}$$

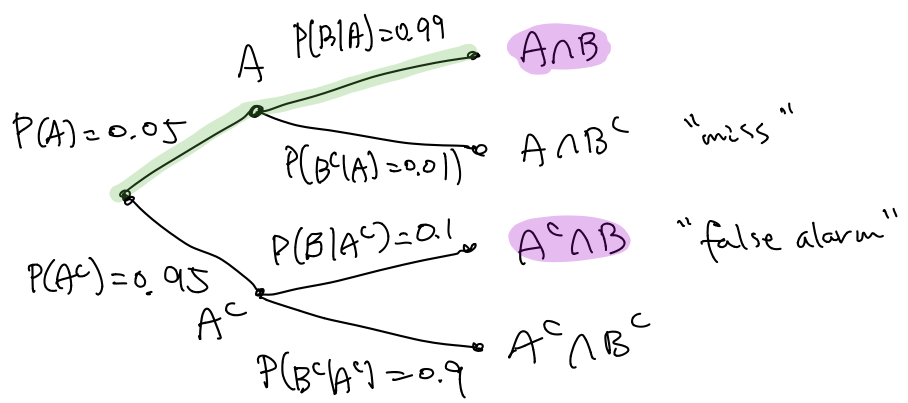
$A \cap B \cap C \cap B = A \cap C \cap B \cap B = \emptyset \cap B \cap B = \emptyset.$

(Same also for finite or countable additivity).

<u>Remark</u> As conditional probability satisfy all the axioms above, any form or theorem we ever derive for probabilities also remain true for conditional probabilities.

2.5 A Radar example & 3 basic tools.

<u>Example</u>. Airplane registering on radar.



Event A = Airplane is flying above

Event B = Something registers on radar.

i) $P(A \cap B) = P(A)P(B|A) = 0.05 \cdot 0.99$

⇝ the multiplication rule

ii) $P(B) = P(A \cap B) + P(A^c \cap B) = P(A)P(B|A) + P(A^c)P(B|A^c)$

$= 0.05 \cdot 0.99 + 0.95 \cdot 0.1$.

⇝ Total probability theorem

iii) $P(A|B)$ = "Airplane is really there when registered on radar"

$= P(A \cap B) / P(B) \simeq 0.34$ ⇝ Not really convincing!

⇝ The Bayes' rule

2.b . The multiplication Rule.

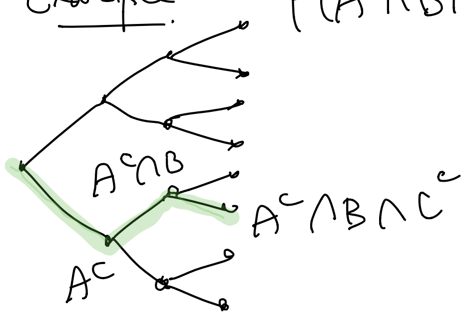
$P(A \cap B) = P(B) \cdot P(A|B)$

- $P(A \cap B)$: Probability of two events to both occur
- $P(B)$: Probability of second event to occur
- $P(A|B)$: Probability of first event to occur given second event occured.

$= P(A) \cdot P(B|A)$

"Free to choose A or B to be the first event".

Example

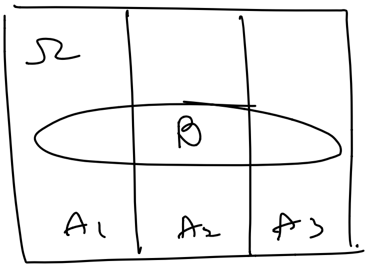


$$P(A^c \cap B \cap C^c) = P((A^c \cap B) \cap C^c)$$
$$= P(A^c \cap B) P(C^c | A^c \cap B)$$
$$= P(A^c) P(B|A^c) P(C^c | A^c \cap B).$$

∴ For n events, $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \prod_{i=2}^{n} P(A_i | \bigcap_{j=1}^{i-1} A_j)$

2.7 Total Probability Theorem

Example. Partition of sample space into A_1, A_2, A_3.



It has $P(A_i)$ and $P(B|A_i)$ for every i.

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$
$$= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$$

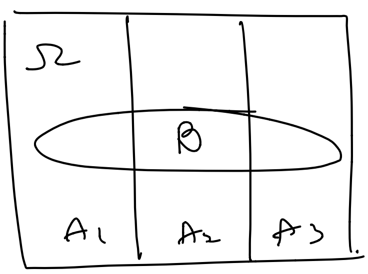
$\therefore$ For n events, $P(B) = \sum_i P(A_i)P(B|A_i)$. if $\sum_i P(A_i) = 1$.

$\Rightarrow$ "Weighted average of $P(B|A_i)$, weighted by $P(A_i)$"

Remark. This thm is also true on infinite countable sets.

2.8 Bayes' Rule.

Example. Partition of sample space into A_1, A_2, A_3.



It has $P(A_i)$ and $P(B|A_i)$ for every i.

$P(A_i)$ → Initial "beliefs"; $P(B|A_i)$ → Additional information

↓

How to revise the "beliefs", given that B occured?

Revised "belief" $P(A_i|B) = \dfrac{P(A_i \cap B)}{P(B)} = \dfrac{P(A_i)P(B|A_i)}{P(B)}$.

$$\therefore P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_j P(A_j)P(B|A_j)}$$

⇝ Systematic approach for incorporating new evidence

* Bayesian Inference

- Initial beliefs $P(A_i)$ on possible causes of an observed event B.

- Model of the world under each A_i : $P(B|A_i)$.

$$A_i \xrightarrow[P(B|A_i)]{\text{model}} B$$

"How likely B is going to occur in each particular situation A_i."

- Draw conclusions from causes

$$B \xrightarrow[P(A_i|B)]{\text{inference}} A_i$$

"How likely a particular scenario A_i is to occur event B"