Lecture 3. Independence

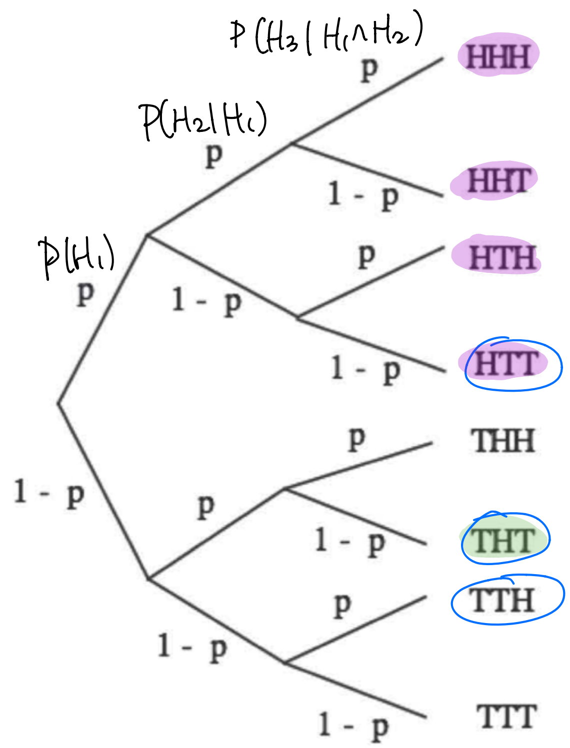
3.1 Lecture Overview.

" If we get to know that some event is occured, usually it changes the probability of the next event. But if the conditional probability and unconditional probability are the same, we call these two events independent: as event A does not carry any useful info regard to B. "

3.2 A coin tossing example

: A model based on conditional probabilities.

* 3 tosses of a biased coin ($P(H) = p$, $P(T) = 1-p$)



i) $P(THT) = P(T_1) \cdot P(H_2|T_1) \cdot P(T_3|H_2 \cap T_1)$
$= (1-p) \cdot p \cdot (1-p)$

↳ Multiplication Rule

ii) $P(1\ H) = P(HTT) + P(THT) + P(TTH)$
$= 3p(1-p)^2$.

↳ Total probability

iii) $P(H_1 | 1H) = \dfrac{P(H_1 \cap 1H)}{P(1H)} = \dfrac{p(1-p)^2}{3p(1-p)^2} = \dfrac{1}{3}$.

$P(H_1 \cap 1H) \rightarrow P(HTT)$

↳ Bayes' Rule.

Remark. $P(H_2 | H_1) = P(H_2 | T_1) = p$. $P(H_2) = P(T_1)P(H_2|T_1) + P(H_1)P(H_2|H_1) = p$.

Probability of second toss doesn't change no matter what first toss was.

3.3 Independence of two events.

* Intuitive approach to independence.

Occurence of A provides no new information about B.

$\Rightarrow$ New knowledge about A doesn't affect B.

$\Rightarrow P(B|A) = P(B) \rightsquigarrow P(A\cap B) = P(B|A)\cdot P(A) = P(B)P(A)$

Def. Event A and B are independent if $P(A\cap B) = P(A)P(B)$.

Remark. Why use this definition rather than an intuitive one?

i) It covers the intuitive one. pf) $P(B|A) = \dfrac{P(B\cap A)}{P(A)} = \dfrac{P(A)P(B)}{P(A)} = P(B)$. $\blacksquare$

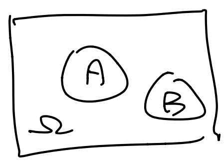
ii) It is symmetric with respect to the roles of A and B.

i.e., B is independent regard to A $\Rightarrow$ A and B are independent.

pf) $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{P(A)P(B)}{P(B)} = P(A)$

iii) It is applicable even if $P(A)=0$ or $P(B)=0$.

Example. If $P(A)>0$, $P(B)>0$, $P(A\cap B)=0$, (A and B is disjoint)



then A and B is not independent.

Remark. Intuitively, these two events are dependent.

If you know A occured, you automatically know that B didn't occur.

Remark. Typically, we have independence when the occurence of each of the two events are determined by two physically distinct and non-interacting processes.

3.4 Independence of event complements

Thm. If A and B are independent, then A and B^c are independent.

Remark. Intuitively, if the new information that A has occurred doesn't change the beliefs on the likelihood of B occurring, it shouldn't also change the likelihood of B not occurring.

pf) $A = (A \cap B) \cup (A \cap B^c)$.

$$P(A) = P(A \cap B) + P(A \cap B^c) = P(A)P(B) + P(A \cap B^c).$$

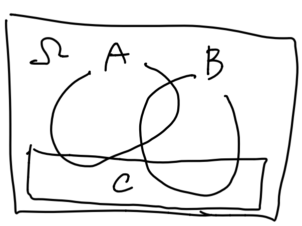
$$P(A \cap B^c) = P(A)(1 - P(B)) = P(A)P(B^c). \quad \boxtimes$$

3.5 Conditional Independence.

Def. Conditional independence given C is defined as independence under the probability law $P(\cdot \mid C)$.

A and B are conditionally independent given C if $P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$.

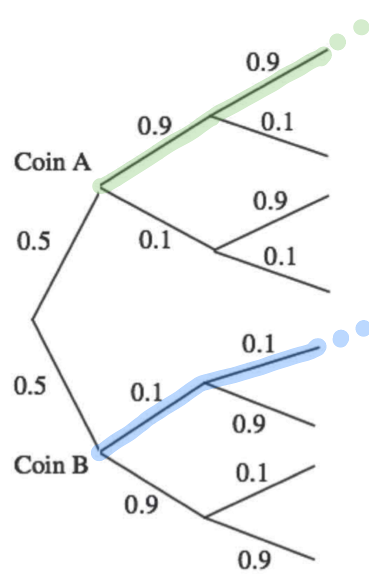
Example



Assume A and B are independent.
But A and B are disjoint given C.
So A and B are not conditionally independent given C.

3.6 Conditioning may effect independence.

Example. Two unfair coins: $P(H \mid A) = 0.9$, $P(H \mid B) = 0.1$.
Are coin tosses independent?



$$P(H_{11}) = P(A)P(H_{11} \mid A) + P(B)P(H_{11} \mid B) = 0.5 \cdot 0.9 + 0.5 \cdot 0.1 = 0.5.$$

$$P\left(H_{11} \,\middle|\, \bigcap_{i=1}^{10} H_i\right) \approx P(H_{11} \mid A) = 0.9.$$

Intuition: It is far more likely to be A rather than B.

3.7 Independence of a collection of events

* Intuitive approach to independence of multiple events.

In a fair coin flip, no matter how many times you flip, you can't obtain more information on the next flip.

$\Rightarrow$ Coin tosses are "independent".

$\Rightarrow$ Information on some of the events does not change probabilities related to the remaining events.

$\Rightarrow$ for $A_1, A_2, \cdots, A_n$, divide events into two groups I_1, I_2.

Let B_i = Any set operations done with events $A \in I_i$.

Then, if $A_1, A_2, \cdots, A_n$ are independent, $P(B_1) = P(B_1 | B_2)$.

<u>Def</u>. Events $A_1, A_2, \cdots, A_n$ are called independent

if $P(A_i \cap A_j \cap \cdots \cap A_m) = P(A_i)P(A_j)\cdots P(A_m)$ for any distant indices $i, j, \cdots m$.

For all possible choices

For any number of events involved.

<u>Example</u>. $n=3$. $\Rightarrow$

$$\begin{cases} \left.\begin{array}{l} P(A_1 \cap A_2) = P(A_1)P(A_2) \\ P(A_1 \cap A_3) = P(A_1)P(A_3) \\ P(A_2 \cap A_3) = P(A_2)P(A_3) \end{array}\right\} \text{Pairwise independence.} \\ P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3). \end{cases}$$

3.8 Independence vs. Pairwise Independence.

<u>Example</u>. Two independent fair coin tosses.

Let $C = \{\text{two tosses have same result}\} = \{H_1H_2, T_1T_2\}$.

$P(H_1) = P(\{H_1T_2, H_1H_2\}) = 1/2$. $P(H_2) = P(\{H_1H_2, T_1H_2\}) = 1/2$. $P(C) = 1/2$.

Pairwise independance.
$$\begin{cases} P(H_1 \cap C) = P(\{H_1H_2\}) = 1/4 = P(H_1)P(C). \Rightarrow H_1 \text{ and } C \text{ are independent.} \\ P(H_2 \cap C) = P(\{H_1H_2\}) = 1/4 = P(H_2)P(C). \Rightarrow H_2 \text{ and } C \text{ are independent.} \\ P(H_1 \cap H_2) = P(\{H_1H_2\}) = 1/4 = P(H_1)P(H_2). \Rightarrow H_1 \text{ and } H_2 \text{ are independent.} \end{cases}$$

$P(C \cap H_1 \cap H_2) = P(\{H_1H_2\}) = 1/4 \neq P(C)P(H_1)P(H_2)$. $\Rightarrow C, H_1, H_2$ are <u>not</u> independent.

$\Leftrightarrow P(C|H_1) = P(C|H_2) = P(C)$, but $P(C|\underline{H_1 \cap H_2}) \neq P(C)$.

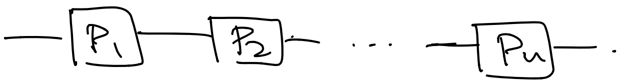
Additional information if both H_1 and H_2 occur.

3.9 Example: Reliability of a system

Let U_i : ith unit is up. $P(U_i) = p_i$.

F_i : ith unit is down. $P(F_i) = P(U_i^c) = 1 - p_i$.

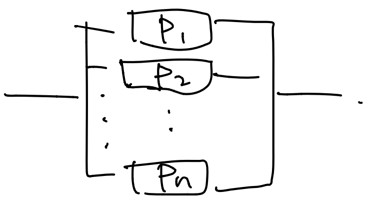
i) Series connection of a system.



All units are okay

$$P(\text{System is up}) = P\left(\bigcap_{i=1}^{n} U_i\right) = \prod_{i=1}^{n} p_i .$$

ii) Parallel connection of a system.



All units are down

$$P(\text{System is up}) = P\left(\bigcup_{i=1}^{n} U_i\right) = 1 - P\left(\bigcap_{i=1}^{n} F_i\right) = 1 - \prod_{i=1}^{n} (1 - p_i)$$

3.10 Example: "The king's sibling" problem.

Q. The king comes from a family of two children.
What is the probability that his sibling is female?

Assumptions: Male has precedence. $P(\text{Male}) = P(\text{Female}) = 1/2$.

A. $$P(\text{Sibling is female} \mid \text{One of two children is male}) = \frac{P(\{MF, FM\})}{P(\{MM, MF, FM\})} = \frac{2}{3} \;\blacksquare$$

But this is correct only if number of siblings was predetermined regardless of birth.
So, with additional scenarios (modeling), probabilities change.

Q. What if they gave birth till 1 male was born? $\dfrac{P(\{FM\})}{P(\{FM\})} = 1 \;\blacksquare$

Q. What if they gave birth till 2 male were born? $\dfrac{P(\{\})}{P(\{MM\})} = 0 \;\blacksquare$

Remark When we deal with situations described in words, somewhat vaguely, we must be very careful to state whatever assumptions are made, and that has to be done before we choose a particular model.