Lecture 4. Counting

4.1 Lecture Overview.

"Developing methods of counting the number of elements in an implicitly described set to find out probabilities of discrete sets"

4.2-3 Basic Counting principle.

Example. 4 shirts. 3 ties. 2 jackets. Number of possible attires?

$4 \times 3 \times 2 = 24$.

→ Generalizing this example,

fixed number of choices, regardless of choices before.

for r stages, n_i choices at stage i,

number of choices is $\prod_{i=1}^{r} n_i$ ⟹ Counting principle.

Example. Permutations (Number of ways of ordering n elements.)

$\underbrace{n}_{\text{1st}} \cdot \underbrace{(n-1)}_{\text{2nd}} \cdot \underbrace{(n-2)}_{\text{3rd}} \cdots \underbrace{1}_{\text{last element}} = n!$

choices for 1st 2nd 3rd ··· last element.

Example. Number of subsets for $\{1, 2, \cdots, n\}$.

$\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{\text{for } n \text{ elements}} = 2^n$

exist or not

for n elements.

Example. Die Roll Example.

Find the probability that six rolls of a six-sided die all give different number.

Assume all outcomes equally likely. A?

$$P(A) = \frac{\text{Number of elements in } A}{\text{Number of possible outcomes}} = \frac{6!}{6^6}$$

$6!$ ← permutation.

6^6 ← 6 choice per roll.

4.4 Combinations

Def. $\binom{n}{k}$ = number of k-element subset of a given n-element set.

Example. Constructing an ordered sequence of k distinct items

i) By choosing k items one at a time.

$$n\cdot(n-1)\cdot(n-2)\cdots(n-k+1) = n!/(n-k)!$$

Choosing the 1st 2nd 3rd ... kth item from $\{n, n-1, \cdots, 1\}$.

ii) By choosing k items, then order them.

$$\underset{\substack{\text{Choosing } k\\ \text{from } n}}{\binom{n}{k}} \cdot \underbrace{k\cdot(k-1)\cdots 1}_{\text{order } k \text{ items.}} = \binom{n}{k}\cdot k!$$

$$\therefore \frac{n!}{(n-k)!} = \binom{n}{k} k!$$

$$\boxed{\therefore \binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (n=0,1,\cdots,k=0,1,\cdots,n)}$$

Example. Extreme cases of $\binom{n}{k}$.

If we adopt the convention $0! = 1$.

i) $\binom{n}{n}$ = choosing every element = only 1 way = $\dfrac{n!}{n!0!} = 1$.

ii) $\binom{n}{0}$ = choosing 0 element = ϕ = only 1 way = $\dfrac{n!}{0!n!} = 1$.

iii) $\sum_{k=0}^{n} \binom{n}{k}$ = Every possible subset of n-element set = 2^n.

4.5 Binomial Probabilities.

Example. For $n \geq 1$ independent coin tosses with $P(H)=p$, $p(k \text{ heads}) = ?$

Example For $n=6$, $P(HTTHHH) = p(1-p)(1-p)ppp = p^4(1-p)^2$.

$$P(\text{Particular sequence}) = p^{\text{number of heads}} (1-p)^{\text{number of tails}}$$

$$P(\text{Particular } k\text{-head sequence}) = p^k (1-p)^{n-k}$$

$\therefore p(k \text{ heads}) = P(\text{Particular } k\text{-head sequence}) \cdot \text{Number of } k\text{-head sequence.}$

$$= \boxed{p^k(1-p)^{n-k} \cdot \binom{n}{k} \quad (\text{Binomial Coefficient})}$$

4.6 A coin tossing problem.

Example. For $n \geq 1$ independent coin tosses with $P(H) = p$,

Given that there were 3 heads in 10 tosses, $\to B$

what is the probability that the first 2 tosses were heads? $\to A$

i) First Solution.

"independence"

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(H_1 H_2 \text{ and one } H \text{ in tosses } 3, 4, \cdots, 10)}{P(B)}$$

$$= \frac{p^2 \cdot P(k=1 \text{ heads for } n=8)}{P(k=3 \text{ heads for } n=10)} = \frac{p^2 \cdot \binom{8}{1} p^1 (1-p)^7}{\binom{10}{3} p^3 (1-p)^7} = \frac{1}{15} \quad \blacksquare.$$

ii) Second Solution.

Every element inside B has the same probability $p^3(1-p)^7$.

$\Rightarrow$ Conditional probability law on B is uniform.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\binom{8}{1}}{\binom{10}{3}} = \frac{1}{15} \quad \blacksquare.$$

4.7-8 Partitions

Example. $n \geq 1$ distinct items, $r \geq 1$ persons give n_i items to person i.

$n_1, n_2, \cdots, n_r$ is a nonnegative integer, with $\sum_{i=1}^{r} n_i = n$.

Let number of choices for partitioning n items into $n_1, n_2, \cdots, n_r$, C.

2-staged process.

Ordering n items $n! =$ Deal n_i to each person i, and then order

$= C \cdot n_1!\, n_2! \cdots n_r!$

$$\therefore \boxed{C = \frac{n!}{n_1!\, n_2! \cdots n_r!} \quad \text{(Multinomial Coefficient)}.}$$

Remark. $r = 2$, $n_1 = k$, $n_2 = n-k$. $\Rightarrow C = \frac{n!}{k!(n-k)!} \Rightarrow$ Binomial coefficient.

Example. 52 card deck, dealt fairly to 4 players. P(Each player gets an ace) = ?

i) First solution

Note that outcomes are partitions, and all partitions are equally likely.

So, the number of total outcomes $= 52!/(13!)^4$

The number of outcomes in given situation

Uniform Probability

$=$ (Distributing the aces) $\cdot$ (Distributing the remaining 48 cards)   2-staged process

$=$ $4!$ $\cdot$ $48!/(12!)^4$

$\therefore$ P(Each player gets an ace) $= \dfrac{4!\,48!/(12!)^4}{52!/(13!)^4}$

ii) Second Solution.

Distribute the aces first. → Is it fair?

All permutations are equally likely, so all partitions are equally likely. So it is fair.

$\therefore \dfrac{52}{52} \cdot \dfrac{39}{51} \cdot \dfrac{26}{50} \cdot \dfrac{13}{49}$

Probability of 1st 2nd 3rd 4th ace to be located correctly.

4.9 Multinomial Probabilities

n trials of r possible outcomes with probability p_i

Example. Balls of different colors $i=1,2,\cdots,r$. Probability of picking ith color is p_i.

Draw n balls independently, given nonnegative integers n_i, with $n_1+n_2+\cdots+n_r = n$.

$P(n_1$ balls of color 1, n_2 balls of color 2, $\cdots$, n_r balls of color $r) = ?$

Example $r=2$, $p_2 = 1-p_1$ (coin flip). $P = \binom{n}{k} p^k (1-p)^{n-k} = \dfrac{n!}{n_1!\,n_2!} p_1^{n_1} p_2^{n_2}$

Example $n=7$, $r=3$. Sequence: 1 1 3 1 2 2 1.

"type" of this sequence: $(4,2,1)$, Probability : $p_1p_1p_3p_1p_2p_2p_1 = p_1^4 p_2^2 p_3^1$

P (Particular sequence of "type" $(n_1, n_2, \cdots, n_r)$) $= p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$

Sequence of type $(n_1, n_2, \cdots, n_r)$ = Partition of $\{1,2,\cdots,n\}$ into subsets of size $n_1, n_2, \cdots, n_r$.

$$\boxed{\therefore P = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r} \cdot \frac{n!}{n_1!\,n_2!\cdots n_r!} \quad \text{(Multinomial Probability)}}$$